# Self-Dual Plane Curves of the Fourth Order.

By L. E. Wear.

## § 1. *Introduction.*

The reciprocal, $r^m$, of a plane rational curve, $\rho^n$, may be regarded as obtained by a polarity which sends any point of $\rho^n$ into a line of $r^m$ and conversely. The singularities of $\rho^n$ will go over into their dual singularities on $r^m$. Now the reciprocal is, in general, distinct from the point-curve and $m \neq n$. The question naturally rises as to when will the two curves coincide. Curves having this property may be called self-dual.* It is evident that for curves of this kind there must be a one-to-one correspondence between the singularities of $\rho^n$ and $r^m$. In other words the order and class must be the same and we have a necessary condition for self-duality expressed by the equation,

$$n = n(n-1) - 2d - 3c, \text{ i.e. } 2d + 3c = n(n-2).$$

In this paper the quartic curve is to be considered and the equation becomes, for $n = 4$,

$$2d + 3c = 8.$$

There are two solutions for this equation, viz.:

$$\text{I.} \quad d = 1, \; c = 2$$
$$\text{II.} \quad d = 4, \; c = 0$$

These are, respectively, the limaçon and the degenerate case of two conics. They will be considered in this order.

## Part I. The Limaçon.

### § 2. *The Equation of the curve.*

The curve is symmetrical with respect to an axis which cuts it in the double point and in two other points which are the vertices of the curve. If we take, as triangle of reference, the axis of the curve and the tangents at the vertices, then the equation is

$$x_0 = at^4 - (a+2)t^2, \; x_1 = (a-2)t^2 - a, \; x_2 = (a-1)t^3 - (a+1)t. \quad (1)$$

* Appel, in *Nouvelles Annales de Math.*, XIII, p. 207, calls such curves "autopolaire." In that article he considered the problem of finding curves self-polar with regard to a given conic—the reverse of the present problem.

The Jacobians of these, two at a time, give the line equation of the curve, which is

$$\begin{aligned}\xi_0 &= (a-1)(a-2)\tau^2 - a(a+1),\\ \xi_1 &= a(a-1)\tau^4 - (a+1)(a+2)\tau^2,\\ \xi_2 &= 2a(2-a)\tau^3 + 2a(2+a)\tau.\end{aligned} \tag{2}$$

That equations (1) are the equations of a limaçon may be seen as follows:

The curve is reflected into itself in the axis $x_2 = 0$, the reflexion being effected by the transformation,

$$t + t' = 0,$$

of which $t = 0$, $t = \infty$ are the double elements. These are the vertices of the curve. In addition to these two points, the line $x_2 = 0$ cuts out the parameters $t^2 = (a+1)/(a-1)$. If these are substituted in the equations of the curve they give only one point, which must, then, be a double point of the curve.

The fundamental involution,* i. e. a pencil of binary forms apolar to each of (1), is

$$\begin{aligned}&(a-2)t^4 + 4(a-1)t^3 + 6at^2 + 4(a+1)t + (a+2)\\ &+ \lambda\,[(a-2)t^4 - 4(a-1)t^3\\ &+ 6at^2 - 4(a+1)t + (a+2)]\end{aligned} \tag{3}$$

If we substitute the coefficients of this pencil in the condition for a cusp which is given by Professor Morley in his "Notes on Projective Geometry," p. 40, we find that the condition is satisfied, and, hence, that the curve has a cusp. Since the curve is reflected into itself in $x_2 = 0$, and since the cusp does not lie on the axis then there must be a second one, the reflexion of the first. The curve is, therefore, one having a double point and two cusps, i. e., is the limaçon.

The two cusps are given by $t^2 = 1$. The flexes are given by the Jacobian of the two members of the fundamental involution,† and are

$$t^2 = (a+1)(a+2)/(a-1)(a-2).$$

## § 3. *Polarities.*

Now any correlation which sends the curve into itself must interchange cusps and flexes. Hence there may be two such correlations corresponding to

* See a paper by Stahl, *Crelle*, Vol. 101, p. 300.

† Meyer: *Apolarität und Rationale Kurven*, p. 244.

the two ways in which the cusps and flexes may be paired. These two correlations are given by the transformations

$$t\tau = \sqrt{(a+1)(a+2)/(a-1)(a-2)} \tag{4}$$

and

$$t\tau = -\sqrt{(a+1)(a+2)/(a-1)(a-2)} \tag{5}$$

We require now that (4) and (5) shall send any point of the limaçon into a line of the curve and conversely. In order to do this operate with (4) or (5) on the equation of a point and identify the resulting expression in the parameter with the equation of a line. Thus will the parameter of a point of the curve be interchanged with that of a line of the curve and conversely, and hence there will be obtained a correlation which sends the curve into itself.

Now if we substitute in the incidence condition

$$(x\xi) = 0,$$

of point and line, the coördinates $x_i$ from equations (1), we have the equation of any point of the limaçon; likewise, if we substitute the coördinates $\xi_i$ from equations (2) we have the equation of any line tangent to the curve. Making these substitutions we find as the equations of point and line respectively,

$$\begin{aligned}&[at^4-(a+2)t^2]\,\xi_0+[(a-2)t^2-a]\,\xi_1\\&+[(a-1)t^3-(a+1)t]\,\xi_2=0\end{aligned} \tag{6}$$

and

$$\begin{aligned}&[(a-1)(a-2)\tau^2-a(a+1)]\,x_0+[a(a-1)\tau^4(a+1)(a+2)\tau^2]x_1\\&+[2a(2-a)\tau^3+2a(2+a)\tau]x_2=0\end{aligned} \tag{7}$$

If we now make the transformation (4) in equation (7), i. e. put $\tau=(1/t)\sqrt{(a+1)(a+2)/(a-1)(a-2)}$, clear the resulting equation of fractions and remove the factor $(a-1)$, the result is

$$\begin{aligned}&(a^2-1)(a-2)^2\sqrt{(a-1)(a-2)}\;[at^4-(a+2)t^2]x_0\\&+(a+1)^2(a+2)^2\sqrt{(a-1)(a-2)}\;[(a-2)t^2-a]x_1\\&-2a(a+2)(a-2)^2\sqrt{(a+1)(a+2)}\;[(a-1)t^3-(a+1)t]x_2=0.\end{aligned} \tag{8}$$

Since this is a line on the point $t$ of the curve, it is identical with equation (6), regarding both equations as functions of the parameter. Making this identification, we have

$$\begin{aligned}-a\xi_0&=a(a^2-1)(a-2)^2\sqrt{(a-1)(a-2)}\;x_0,\\a\xi_1&=-a(a+1)^2(a+2)^2\sqrt{(a-1)(a-2)}\;x_1,\\(a+1)\xi_2&=2a(a+1)(a+2)(a-2)^2\sqrt{(a+1)(a+2)}\;x_2,\end{aligned}$$

wherein we have equated the coefficients of $t^4$, $t^0$ and $t$ respectively. Dividing the first equation by $-a$, the second by $a$ and the third by $a+1$, the equations become finally,

$$\left.\begin{aligned} \xi_0 &= (1-a^2)(a-2)^2\sqrt{(a-1)(a-2)}\; x_0, \\ \xi_1 &= -(a+1)^2(a+2)^2\sqrt{(a-1)(a-2)}\; x_1, \\ \xi_2 &= 2a(a+1)(a+2)(a-2)^2\sqrt{(a+1)(a+2)}\; x_2, \end{aligned}\right\} \tag{9}$$

If the transformation

$$\tau = -(1/t)\sqrt{(a+1)(a+2)/(a-1)(a-2)}$$

had been made in the equation of a line, then only $\xi_2$ will be changed in (9), since it comes from odd-powered terms, while $\xi_0$ and $\xi_1$ come from even powers. Furthermore the coördinate $\xi_2$ will be changed only in sign. Hence there arises from the second transformation the correlation

$$\left.\begin{aligned} \xi_0 &= (1-a^2)(a-2)^2\sqrt{(a-1)(a-2)}\; x_0, \\ \xi_1 &= -(a+1)^2(a+2)^2\sqrt{(a-1)(a-2)}\; x_1, \\ \xi_2 &= -2a(a+1)(a+2)(a-2)^2\sqrt{(a+1)(a+2)}\; x_2, \end{aligned}\right\} \tag{10}$$

Equations (9) and (10) are two correlations which send any point of the curve into a line of the curve, i. e. send the curve into itself. In particular dual singularities are interchanged, as may be easily verified.

Furthermore if we examine the determinants of equations (9) and (10) we find them to be of the form

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix},$$

i. e. the two correlations are actually polarities and the conics giving them are

$$\begin{aligned} &(1-a^2)(a-2)^2\sqrt{(a-1)(a-2)}\; x_0^2 \\ &-(a+1)^2(a+2)^2\sqrt{(a-1)(a-2)}\; x_1^2 \\ &\pm 2a(a+2)(a-2)^2\sqrt{(a+1)(a+2)}\; x_2^2 = 0. \end{aligned} \tag{11}$$

These conics are conjugate, in the sense that they have double contact and are reflected, the one into the other, by a pair of reflexions in the reference triangle. It follows that each conic is its own polar reciprocal as to the other. This fact is seen from a geometrical viewpoint, since a polarity leaving the curve unaltered must also leave the conic of the other polarity unaltered.

The points of contact of the two conics lie on the axis $x_2 = 0$. This can be proved from general considerations as follows: Call the vertices of the

limaçon $A$ and $A'$ (See Fig. 1); the polar of $A$ ($t=\infty$) is the tangent at $A'(t=0)$ and conversely. Hence $A$ and $A'$ are a pair harmonic to the meets of $x_2=0$ with both of the conics. Also the polar of the double point $D$ is the double line which cuts the axis at $D'$, say. Then $D$ and $D'$ are also a pair of the involution on the line and the Jacobian of the two pairs $(A, A')$ and $(D, D')$ will give the points $(C, C')$ where the conics cut $x_2=0$ and where they have contact,—since the polars of those points are tangent to both conics at the poles themselves. Further, the line $x_2=0$ is an axis of each one of the conics, since the polars of points on it, $(A, A', D, D')$, are lines perpendicular to the axis of the limaçon, and the pole of the latter is a point at infinity where any two of these perpendiculars intersect.

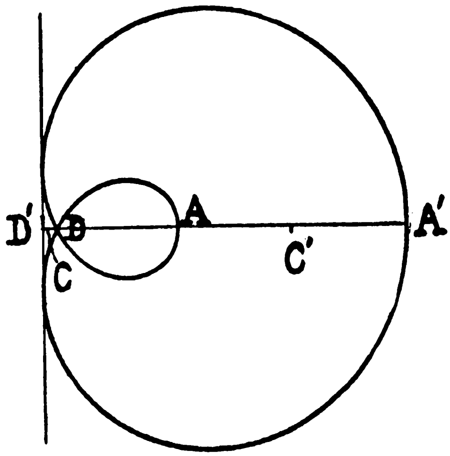


FIG. 1.

There will be certain points on the curve which will be fixed under the polarities, i. e. are transformed into tangents at the same points. These fixed points are four in number and are found by letting $t$ and $\tau$ come together in equations (4) and (5). They are given by

$$t^4=(a+1)(a+2)/(a-1)(a-2).$$

Since at these four points the polar of $t$ is a tangent to the curve at $t$, therefore each of the conics (11) has double contact with the limaçon. One conic has real contacts with the curve and the other has imaginary contacts.

## § 4. *Special Cases.*

Conics (11) will degenerate when their discriminants vanish. The latter are $\pm 2a(a-1)(a+1)^4(a-2)^5(a+2)^5\sqrt{(a+1)(a+2)}$ which vanish for the values $a=\pm 1, \pm 2$. But equations (11) vanish identically

for $a = -1$ and $a = +2$. Since the flexes are given by

$$t^2 = (a+1)(a+2)/(a-1)(a-2),$$

the value $a = +1$ signifies that two flexes have united at $t = \infty$ and it may be easily verified that they unite to form a third cusp. The cusp-tangent is $x_2 = 0$, which, taken twice, is the equation of the conics for the value $a = -1$. For $a = -2$, two flexes of the curve unite but in this case they form an undulation point, i. e. a point where the tangent to the curve meets the curve

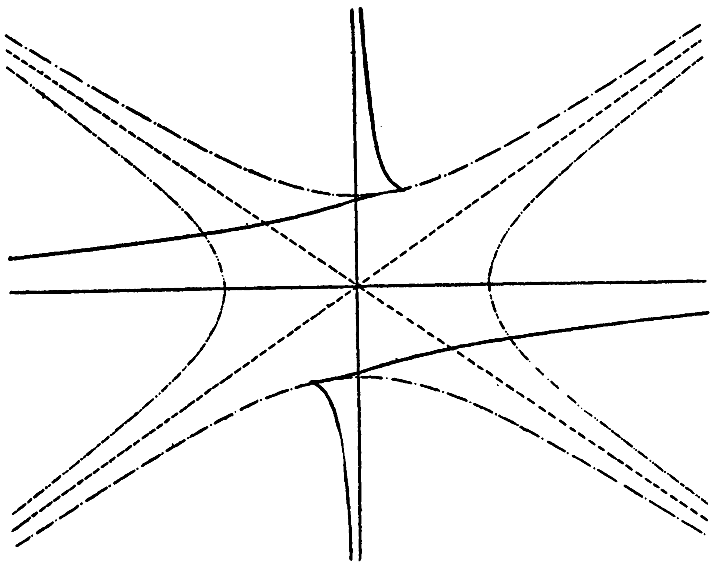

FIG. 2.

in four coincident points. The conics, for $a = -2$, degenerate into the line $x_0 = 0$, which is the equation of the undulation tangent.

## § 5. *Summary.*

We have seen that the limaçon admits of a reflexion given by the equation

$$t + t' = 0,$$

and is invariant under the two polarities $\pi_0$ and $\pi_1$. Call the reflexion $R$. Since $R^2 = 1$, the elements 1, $R$, form a group $(G_2)$ of collineations under which the curve is invariant. It is evident also that

$$\pi_0 R = \pi_1 \text{ and } \pi_1 R = \pi_0.$$

Further

$$\pi_0\pi_1 = R \text{ and } \pi_0^2 = \pi_1^2 = 1.$$

Hence the

Theorem: *The limaçon is invariant under a $G_4$ consisting of two collineations and two polarities.*

Furthermore all possible polarities and correlations which leave the curve fixed are exhausted in $\pi_0$ and $\pi_1$. For, suppose another to exist—say $\pi_m$. Then either

$$\pi_0\pi_m = 1 \text{ or } \pi_0\pi_m = R$$

In the first case

$$\pi_m = \pi_0,$$

and in the second

$$\pi_m = \pi_1.$$

In Figure 2 are shown the limaçon (in Cartesian coördinates), with the two conics, for the case $a = \frac{1}{2}$.

## § 6. *Satellite Conic.*

Of some interest in connecting with the plane quartic curve is the Satellite Conic of a line. In the case of the cubic curve the corresponding thing is the Satellite Line. Any line, $\xi$, will cut the cubic in three points. The tangents to the curve at these three points meet the curve again in three other points which lie on a line,* called the Satellite Line of $\xi$. In the case of the plane quartic a line, $\xi$, will cut the curve in four points $T_i$. The tangents to the curve at these four points meet the curve again in eight points which lie on a conic,† the Satellite Conic of the line $\xi$. The problem here is to find the actual equation of this conic for the limaçon.

The condition that three points of the plane rational quartic be on a line is as follows:

$$\begin{aligned}&p_{01}S_3^2 + p_{02}S_2S_3 + (p_{03} - p_{12})S_1S_3 + p_{12}S_2^2 + (p_{04} - p_{13})S_3\\&+ p_{13}S_1S_2 + (p_{14} - p_{23})S_2 + p_{23}S_1^2 + p_{24}S_1 + p_{34} = 0.\end{aligned}$$

The $p_{ij}$ refer to the determinants $\begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}$ of the fundamental involution $(at)^4 + \lambda(bt)^4$ (See Art. 2), and the $S_i$ are symmetric functions of the three parameters of the points. Substituting the appropriate values of the $p_{ij}$ from

* Salmon, *Higher Plane Curves*, Art. 179.

† Salmon, *l. c.*, Art. 30.

the fundamental involution of Art. 2, we have, as the condition that three points of the limaçon be collinear,

$$-(a-1)(a-2)S_3^2-2(a^2-a-1)S_1S_3+a(a-1)S_2^2$$
$$+2(a^2+a-1)S_2-a(a+1)S_1^2+(a+1)(a+2)=0.$$

Herein set $t_1=t_2=T$, $t_3=t$ and we have

$$\begin{aligned}&-(a-1)(a-2)t^2T^4-2(a^2-a-1)(t+2T)(tT^2)\\&+a(a-1)(2tT+T^2)^2+2(a^2+a-1)(2tT+T^2)\\&-a(a+1)(t+2T)^2+(a+1)(a+2)=0.\end{aligned}\tag{13}$$

This is a relation, $f(T^4, t^2)=0$, connecting $T$, the point of tangency of a line, and the two remaining points of intersection, $t$. It says that, given the line $T$, the two remaining points of intersection with the curve are given by (13). Conversely, given any point $t$ on the curve, there are four tangents, $T$, to the curve from this point, given by (13). Since *any* line $T$, on one of the cusps will be incident with a $t$, then $(T^2-1)$ must be a factor of equation (13). Rearranging the latter in powers of $T$ we have

$$\begin{aligned}&[(a-1)(a-2)t^2-a(a-1)]T^4-4tT^3-2[(a^2-a+1)t^2\\&-(a^2+a+1)]T^2+4tT+[a(a+1)t^2-(a+1)(a+2)]=0,\end{aligned}\tag{13'}$$

and dividing by $(T^2-1)$ we have, finally,

$$\begin{aligned}&[(a-1)(a-2)t^2-a(a-1)]T^2-4tT\\&-[a(a+1)t^2-(a+1)(a+2)]=0.\end{aligned}\tag{13''}$$

We have here a relation, $f(T^2, t^2)=0$, connecting the point $t$ and the line $T$ of the limaçon. For the self-dual quartic, then, on a line $T$ are two points $t$, and from a point $t$ are two tangents $T$, i. e. the relation is a perfectly symmetric one. It is invariant under the reflexion, $t+t'=0$, and also under the transformations (4) and (5). Letting $t=T$, we have a quartic giving the four points of the curve where a point $t$ is coincident with a point of tangency, $T$. This quartic is

$$(a-1)(a-2)T^4-2(a^2+2)T^2+(a+1)(a+2)=0,$$

which factors into,

$$(T^2-1)[(a-1)(a-2)T^2-(a+1)(a+2)]=0.$$

These two factors give the cusps and flexes, respectively. Evidently at these points the tangent lines have three coincident intersections with the curve. The incidence condition of point and line is $(x\xi)=0$. Put therein the values

of $x_i$ from equations (1), obtaining

$$[aT^4-(a+2)T^2]\,\xi_0+[(a-2)T^2-a]\,\xi_1 \\ +[(a-1)T^3-(a+1)T]\,\xi_2=0. \tag{14}$$

Arranging in powers of $T$ we have

$$a\xi_0T^4+(a-1)\xi_2T^3-[(a+2)\xi_0-(a-2)\xi_1]T^2 \\ -(a+1)\xi_2T-a\xi_1=0. \tag{14'}$$

Given a line $\xi$, equation (14′) fixes the parameters of the four points of intersection with the curve.

Now let the $T$'s of (14′), be the same as those of (13″); i. e. let the two equations have common roots. The condition that the two have common roots is the vanishing of their eliminant, which will be of the fourth degree in the coefficients of (13″), and of the second degree in those of (14′).* Hence a $f(\xi^2, t^8)=0$, a relation connecting $\xi$ and the eight points $t$ when the points of tangency $T_i$ lie on $\xi$. This eliminant, formed according to Sylvester's dialytic method,† is

$$\begin{aligned}
&a[a(a+1)^2(a^4+4a^3-8a+4)\xi_0^2+2a(a+1)(a-1)(a-2)^2\\
&(a^2+2a-2)\xi_0\xi_1+a(a-1)^2(a-2)^4\xi_1^2-(a+1)^3(a-1)^3\\
&(a-2)\xi_2^2]t^8-8a(a+1)^2(a-1)(2a-1)\xi_0\xi_2t^7-4[a(a+1)^2\\
&(a+2)(a^4+3a^3-4a+2)\xi_0^2+2a(a-2)\\
&(a^6+a^5-5a^4-3a^3+7a^2-2a-1)\xi_0\xi_1+a^2(a-1)^3(a-2)^3\xi_1^2\\
&-(a+1)^2(a-1)^2(a^4-a^3-a^2-a+1)\xi_2^2]t^6+8[2(a+1)^2\\
&(2a-1)(a^2+a-1)\xi_0\xi_2+(a-1)^3(a-2)(2a+1)\xi_1\xi_2]t^5\\
&+2[(a+1)^2(a+2)^2(3a^4+6a^3-4a+2)\xi_0^2+2(3a^8-21a^6\\
&+46a^4-28a^2+8)\xi_0\xi_1+(a-1)^2(a-2)^2(3a^4-6a^3+4a+2)\xi_1^2\\
&-(a+1)^2(a-1)^2(3a^4-3a^2-4)\xi_2^2]t^4-8[(a+1)^3(a+2)\\
&(2a-1)\xi_0\xi_2+2(a-1)^2(2a+1)(a^2-a-1)\xi_1\xi_2]t^3\\
&-4[a^2(a+1)^3(a+2)^3\xi_0^2+2a(a+2)(a^6-a^5-5a^4+3a^3+7a^2+2a-1)\\
&\xi_0\xi_1+a(a-1)^2(a-2)(a^4-3a^3+4a+2)\xi_1^2-(a+1)^2(a-1)^2\\
&(a^4+a^3-a^2+a+1)\xi_2^2]t^2+8a(a+1)(a-1)^2(2a+1)\xi_1\xi_2t\\
&+a[a(a+1)^2(a+2)^4\xi_0^2+2a(a+1)(a-1)(a+2)^2(a^2-2a-2)\\
&\xi_0\xi_1+a(a-1)^2(a^4-4a^3+8a+4)\xi_1^2-(a+1)^3(a-1)^3\\
&(a+2)\xi_2^2]=0.
\end{aligned} \tag{15}$$

Let a line $\xi$ cut the curve in four points $T$. Draw the four tangents at these points. These tangents cut the curve in eight other points the para-

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* See Salmon's Lessons on Higher Algebra, Art. 70.

† Salmon, l. c., Art. 83.

meters of which are given by the octavic (15). The next step is to find the equation of a conic passing through these eight points.

Any conic

$$\sum_{i,k=0}^{2} a_{ik}x_i x_k = 0, \qquad a_{ik} = a_{ki}$$

will cut the curve in eight points which are obtained by substituting in the equation of the conic the values of $x_i$ from equations (1). We have then

$$\begin{aligned}&a_{00}[at^4 - (a+2)t^2]^2 + a_{11}[(a-2)t^2 - a]^2 + a_{22}[(a-1)t^3\\ &- (a+1)t]^2 + 2a_{01}[at^4 - (a+2)t^2][(a-2)t_2 - a] + 2a_{02}[at^4\\ &- (a+2)t^2][(a-1)t^3 - (a+1)t] + 2a_{12}[(a-2)t^2 - a]\\ &[(a-1)t^3 - (a+1)t] = 0.\end{aligned} \tag{16}$$

Simplifying and arranging in powers of $t$ we have

$$\begin{aligned}&a^2a_{00}t^8 + 2a(a-1)a_{02}t^7 + [-2a(a+2)a_{00} + (a-1)^2a_{22}\\ &\qquad\qquad\qquad\qquad + 2a(a-2)a_{01}]t^6\\ &+ [-2a(a+1)a_{02} - 2(a-1)(a+2)a_{02} + 2(a-1)(a-2)a_{12}]t^5\\ &+ [(a+2)^2a_{00} + (a-2)^2a_{11} - 2(a+1)(a-1)a_{22} - 4(a^2-2)a_{01}]t^4\\ &+ [-2(a+1)(a+2)a_{02} - 4(a^2-a-1)a_{12}]t^3\\ &+ [-2a(a-2)a_{11} + (a+1)^2a_{22} + 2a(a+2)a_{01}]t^2\\ &+ 2a(a+1)a_{12}t + a^2a_{11} = 0.\end{aligned} \tag{16'}$$

This octavic in $t$ gives the parameters of the eight points cut out by any conic. By identifying (16′) with the octavic of equation (15), we have more than enough conditions to determine the coefficients $a_{ik}$ of the conic. On identifying the two we have:

Coefficients of $t^8$,

$$\begin{aligned}aa_{00} = {}& a(a+1)^2(a^4+4a^3-8a+4)\xi_0^2 + 2a(a+1)(a-1)(a-2)^2\\ &(a^2+2a-2)\xi_0\xi_1 + a(a-1)^2(a-2)^4\xi_1^2 - (a+1)^3(a-1)^3\\ &(a-2)\xi_2^2;\end{aligned}$$

of $t^7$, $\quad a_{02} = -4(a+1)^2(2a-1)\xi_0\xi_2;$

of $t$, $\quad a_{12} = 4(a-1)^2(2a+1)\xi_1\xi_2;$

of $t^0$, $$\begin{aligned}aa_{11} = {}& a(a+1)^2(a+2)^4\xi_0^2 + 2a(a+1)(a-1)(a+2)^2\\ &(a^2-2a-2)\xi_0\xi_1 + a(a-1)^2(a^4-4a^3+8a+4)\xi_1^2\\ &-(a+1)^3(a-1)^3(a+2)\xi_2^2;\end{aligned}$$

of $t^4$, $$\begin{aligned}&-(a+1)(a-1)a_{22} - 2(a^2-2)a_{01} = 2(a+1)^2(a+2)^2\\ &(a^4+2a^3+2a^2+4)\xi_0^2 + 4(a^8-5a^6+2a^4+18a^2-12)\xi_0\xi_1\\ &+2(a-1)^2(a-2)^2(a^4-2a^3+2a^2-4)\xi_1^2\\ &-2(a+1)^2(a-1)^2(a^4+a^2-4)\xi_2^2;\end{aligned} \tag{17}$$

of $t^2$, $(a+1)^2 a_{22} + 2a(a+2)a_{01} = -2a(a+1)^2(a+2)^3$
$$(a^2+2a+4)\xi_0^2 - 4a(a+2)(a^6-3a^4-4a^3+12a+6)\xi_0\xi_1 - 2a^4(a-1)^2(a-2)^2\xi_1^2 + 2(a+1)^2(a-1)^2 (a^4+2a^3+3a^2+2a-2)\xi_2^2 \tag{18}$$

By multiplying (17) by $(a+1)$ and (18) by $(a-1)$ and adding the resulting equations, we eliminate $a_{22}$ and obtain

$$4a_{01} = 4(a+1)^2(a+2)^2(a^2+2a-2)\xi_0^2 + 8(a^6-9a^4+18a^2-6)\xi_0\xi_1 + 4(a-1)^2(a-2)^2(a^2-2a-2)\xi_1^2 - 4(a+1)^2(a-1)^2(a^2-3)\xi_2^2,$$

or

$$a_{01} = (a+1)^2(a+2)^2(a^2+2a-2)\xi_0^2 + 2(a^6-9a^4+18a^2-6)\xi_0\xi_1 + (a-1)^2(a-2)^2(a^2-2a-2)\xi_1^2 - (a+1)^2(a-1)^2(a^2-3)\xi_2^2.$$

Substituting this value of $a_{01}$ in equation (18) and simplifying, we have

$$(a+1)^2 a_{22} = -4a(a+1)^4(a+2)^3\xi_0^2 - 8a^2(a+1)^2(a+2)(a-2)(a^2-3)\xi_0\xi_1 - 4a(a+1)^2(a-1)^2(a-2)^3\xi_1^2 + 4(a+1)^2(a-1)^2 (a^4+2a^3-2a-1)\xi_2^2$$

Hence,

$$a_{22} = -4a(a+1)^2(a+2)^3\xi_0^2 - 8a^2(a+2)(a-2)(a^2-3)\xi_0\xi_1 - 4a(a-1)^2(a-2)^3\xi_1^2 + 4(a-1)^2(a^4+2a^3-2a-1)\xi_2^2.$$

All the coefficients of the conics are determined. By substituting the proper values in the coefficients of $t^6$ and $t^5$ the conditions arising from those two terms are found to be satisfied by the above values. Hence the coefficients of the conic are the following:

$$\begin{aligned}
aa_{00} &= a(a+1)^2(a^4+4a^3-8a+4)\xi_0^2 + 2a(a+1)(a-1)(a-2)^2 (a^2+2a-2)\xi_0\xi_1 + a(a-1)^2(a-2)^4\xi_1^2 - (a+1)^3(a-1)^3(a-2)\xi_2^2,\\
aa_{02} &= -4a(a+1)^2(2a-1)\xi_0\xi_2,\\
aa_{12} &= 4a(a-1)^2(2a+1)\xi_1\xi_2,\\
aa_{11} &= a(a+1)^2(a+2)^4\xi_0^2 + 2a(a+1)(a-1)(a+2)^2 (a^2-2a-2)\xi_0\xi_1 + a(a-1)^2(a^4-4a^3+8a+4)\xi_1^2\\
&\quad - (a+1)^3(a-1)^3(a+2)\xi_2^2,\\
aa_{01} &= a(a+1)^2(a+2)^2(a^2+2a-2)\xi_0^2 + 2a(a^6-9a^4+18a^2-6)\xi_0\xi_1\\
&\quad + a(a-1)^2(a-2)^2(a^2-2a-2)\xi_1^2 - a(a+1)^2 (a-1)^2(a^2-3)\xi_2^2,\\
aa_{22} &= -4a^2(a+1)^2(a+2)^3\xi_0^2 - 8a^3(a+2)(a-2)(a^2-3)\xi_0\xi_1\\
&\quad - 4a^2(a-1)^2(a-2)^3\xi_1^2 + 4a(a-1)^2(a^4+2a^3-2a-1)\xi_2^2.
\end{aligned}$$

The satellite conic is then as follows:

$$
\begin{aligned}
&[a(a+1)^2(a^4+4a^3-8a+4)\xi_0^2+2a(a+1)(a-1)(a-2)^2\\
&(a^2+2a-2)\xi_0\xi_1+a(a-1)^2(a-2)^4\xi_1^2-(a+1)^3(a-1)^3\\
&\qquad(a-2)\xi_2^2]x_0^2\\
&+[a(a+1)^2(a+2)^4\xi_0^2+2a(a+1)(a-1)(a+2)^2\\
&\qquad(a^2-2a-2)\xi_0\xi_1+a(a-1)^2(a^4-4a^3+8a+4)\xi_1^2\\
&\qquad-(a+1)^3(a-1)^3(a+2)\xi_2^2]x_1^2 \qquad\qquad (19)\\
&-4a[a(a+1)^2(a+2)^3\xi_0^2+2a^2(a+2)(a-2)(a^2-3)\xi_0\xi_1\\
&\qquad+a(a-1)^2(a-2)^3\xi_1^2-(a+1)^3(a-1)^3\xi_2^2]x_2^2\\
&+[8a(a-1)^2(2a+1)\xi_1\xi_2]x_1x_2-[8a(a+1)^2(2a-1)\xi_0\xi_2]x_0x_2\\
&+2[a(a+1)^2(a+2)^2(a^2+2a-2)\xi_0^2+2a(a^6-9a^4+18a^2-6)\\
&\qquad\xi_0\xi_1+a(a-1)^2(a-2)^2(a^2-2a-2)\xi_1^2-a(a+1)^2(a-1)^2\\
&\qquad(a^2-3)\xi_2^2]x_0x_1=0.
\end{aligned}
$$

Given a line $\xi$, cutting the limaçon in four points $T$: the tangents at the points $T$ cut in eight other points which lie on the conic given in equation (19).

Among interesting special cases is that of the line $x_2=0$, the axis of the curve. Here $\xi_0=0$, $\xi_1=0$, $\xi_2=1$. Substituting these values in (19) we have, after simplifying,

$$
\begin{aligned}
&(a+1)^2[(a+1)(a-1)(a-2)x_0^2+2a(a^2-3)x_0x_1+(a+1)\\
&\qquad(a-1)(a+2)x_1^2]-4a(a+1)^3(a-1)x_2^2=0. \qquad (20)
\end{aligned}
$$

This is a conic symmetrical as to the line $x_2=0$ and passing through the double point. The latter fact can be seen from the following considerations: The line $x_2=0$ cuts the curve at the vertices and at the double point. The two tangents at the double point have there three points in common with the curve. Hence, two of the eight points, $t_i$, being at the double point, the Satellite must pass through the latter.

Again the line joining the two cusps is $x_0-x_1=0$. Putting $\xi_0=1$, $\xi_1=-1$, $\xi_2=0$ in (19) and simplifying, we have

$$
\begin{aligned}
(2a-1)^4x_0^2+2(16a^4-8a^2-1)x_0x_1+(2a+1)^4x_1^2&\\
-32a^2(2a+1)x_2^2=0.& \qquad (21)
\end{aligned}
$$

This satellite of the line of cusps goes through the cusps themselves and through the two residual intersections of the cuspidal tangents.

Consider next the Satellite of a line $\xi$ tangent to the curve. Such a line cuts in only three points $T_i$, the point of tangency and two others. The tangent lines at $T_i$ will be the line $\xi$ itself and the two tangents at other two points $T_i$. Call these tangents $\eta$ and $\zeta$. They cut the curve in $t_1$, $t_2$, and

$t_1'$, $t_2'$, respectively. Since $\xi$ itself counts as a tangent at one of the points $T_i$, therefore the Satellite passes through the other points $T_2$, $T_3$. Since $T_2$, $t_1$, $t_2$ are on the conic, and likewise $T_3$, $t_1'$, $t_2'$, therefore the Satellite is composed of the two lines $\eta$ and $\zeta$. Hence, the Satellite conic of a line $\xi$ tangent to the curve is composed of two lines, the tangents at the two points where $\xi$ cuts the curve.

For example, the equation of the double line is

$$(a-2)^2x_0-(a+2)^2x_1=0.$$

Substituting $\xi_0=(a-2)^2$, $\xi_1=-(a+2)^2$, $\xi_2=0$ in equation (19), we find the Satellite of the double line to be

$$(a-2)^4x_0^2-2(a+2)^2(a-2)^2x_0x_1+(a+2)^4x_1^2=0, \tag{22}$$

or

$$[(a-2)^2x_0-(a+2)^2x_1]^2=0. \tag{22'}$$

That is, the Satellite of the double line is the double line itself taken twice.

The equation of the flex tangent at

$$T=\sqrt{(a+1)(a+2)/(a-1)(a-2)} \quad \text{is,}$$

$$(a+1)(a-1)(a-2)^2x_0+(a+1)^2(a+2)^2x_1-2a(a+2)(a-2)^2 \\ \sqrt{(a+1)(a+2)/a-1)(a-2)}\;x_2=0. \tag{23}$$

Setting $\xi_0=(a+1)(a-1)(a-2)^2$, $\xi_1=(a+1)^2(a+2)^2$,
$\xi_2=-2a(a+2)(a-2)^2\sqrt{(a+1)(a+2)/(a-1)(a-2)}$

in (19), we have, as the Satellite,

$$\begin{aligned}(a+1)^2&(a-1)(a-2)^4(a^2+2a-1)x_0^2+2(a+1)(a+2)^2\\ &(a-2)^2(a^4+4a^2-1)x_0x_1+(a+1)^2(a-1)(a+2)^4\\ &(a^2-2a-1)x_1^2-4a^2(a+1)(a+2)^3(a-2)^3x_2^2\\ &-4a\sqrt{(a+1)(a+2)/(a-1)(a-2)}\;[(a-1)(a+2)^3\\ &(a-2)^2(2a+1)x_1x_2-(a+1)(a+2)(a-2)^4\\ &(2a-1)x_0x_2]=0.\end{aligned} \tag{24}$$

In this case there is only one intersection in addition to the flex point, namely

$$t=-\frac{(a-1)}{(a+1)}\sqrt{\frac{(a+1)(a+2)}{(a-1)(a-2)}}.$$

The tangent at the latter point is

$$\begin{aligned}(a+1)&(a-2)^2(a^2+2a-1)x_0+(a-1)(a+2)^2(a^2-2a-1)x_1\\ &+2a(a-1)(a+2)(a-2)^2\sqrt{(a+1)(a+2)/(a-1)(a-2)}\\ &x_2=0.\end{aligned} \tag{25}$$

The Satellite (24) is the product of the flex tangent itself, and the line (25).

Let us consider now the dual idea. From any point $x$ in the plane four tangent lines, $t_i$, can be drawn to the curve, touching at four points. From each of these four points are two tangents, $T_i$, to the curve. By a process exactly analogous to the preceding it can be shown that the eight lines, $T_i$, lie on a conic, the Satellite of $x$. The relation $f(x_i, \xi_i) = 0$ so found is identical with equation (19). To prove this statement it is sufficient to say that, owing to the self-duality of the curve, the relation $f(\xi_i, x_i) = 0$, connecting a line $\xi$ and its Satellite must be identical with the relation $f(x_i, \xi_i) = 0$ connecting a point $x$ and its Satellite, since the two ideas are dual ones. Hence equation (19) has a dual interpretation: Given $\xi$, cutting the curve in four points, $T_i$, it is the equation of a conic on the eight points where tangents at $T_i$ meet the curve again; given $x$, from which are four tangents, $t_i$, to the curve, it is the equation of a conic on the eight tangent lines of the curve drawn from the points of contact of the lines $t_i$.

The center of reflexion, admitted by the curve, furnishes a good illustration. The coördinates of the center (0, 0, 1) substituted in (19) give the Satelllite

$$\begin{aligned} a(a+1)^2(a+2)^3\xi_0^2 + 2a^2(a+2)(a-2)(a^2-3)\xi_0\xi_1 \\ + a(a-1)^2(a-2)^3\xi_1^2 - (a+1)^3(a-1)^3\xi_2^2 = 0. \end{aligned} \tag{26}$$

This conic is on the double line and the four tangents drawn from the two vertices of the curve.

From a point $x$ on the curve are only two tangents to the curve $(t_1, t_2)$ in addition to the tangent at $x$. The lines $t_1$, $t_2$ will touch at $y$ and $z$, say. From each of the points $y$ and $z$ are two tangents to the curve which lie on the Satellite of $x$. Furthermore, since the point $x$ is the point of contact of one of the tangents *from* $x$ (the tangent *at* $x$) therefore $t_1$ and $t_2$ lie on the Satellite. Hence, from each of the points $y$ and $z$ are three tangents to the conic, which must, therefore, break up into the two points $y$ and $z$. Thus, the Satellite of a point $x$ on the curve is composed of two points,—the points of contact of tangents from $x$.

For example the Satellite of the double point, $[(a+1), -(a-1), 0]$, is

$$[(a+1)\xi_0 - (a-1)\xi_1]^2 = 0, \tag{27}$$

i. e., the square of the double point itself.

The coördinates of one of the cusps are 1, 1, 1. If substituted in (19), they give for the Satellite

$$(a+1)^2(a^2+2a-1)\xi_0^2+2(a^4+4a^2-1)\xi_0\xi_1+(a-1)^2 \\ (a^2-2a-1)\xi_1^2-(a+1)^2(a-1)^2\xi_2^2-2(a-1)^2 \\ (2a+1)\xi_1\xi_2+2(a+1)^2(2a-1)\xi_0\xi_2=0. \tag{28}$$

From the cusp there is only one tangent to the curve (in addition to the cuspidal tangent) and this meets the curve at $t=-(a+1)/(a-1)$. The equation of the latter point is

$$(a+1)^2(a^2+2a-1)\xi_0+(a-1)^2(a^2-2a-1)\xi_1 \\ -(a+1)^2(a-1)^2\xi_2=0, \tag{29}$$

and that of the cusp (1, 1, 1), is

$$\xi_0+\xi_1+\xi_2=0. \tag{30}$$

Equations (29) and (30) multiplied together give (28), i. e. the Satellite of the cusp consists of two lines one of which is the cusp tangent.

## PART II. TWO CONICS.

### § 7. *The $G_{24}$ of a Four-Point.*

It was pointed out in the introduction that only two cases of self-dual quartics are possible and we come now to the second case, that of two conics regarded as a degenerate $\rho^4$.

To study the properties of the curve it is necessary to consider the four-point common to the two conics and the group of collineations connected therewith; since, if the pair of conics is to be unaltered by correlations then their common four-point and four-line must be merely interchanged. The pair of conics intersect in the same four points, after being acted upon by the correlations, as they did before. In this sense the common four-point and therefore the common self-conjugate triangle are fixed. We assume then that the two conics have a proper, common, self-conjugate triangle, which is taken as the triangle of reference, and that the four points are in the canonical form $(1, \pm 1, \pm 1)$.

The four-point is invariant under a $G_{24}$ of collineations, consisting of reflexions and collineations of periods three and four. Call the four points by the numerals 1, 2, 3, 4 and indicate by subscripts the interchanges made by the transformations. E. g., the notation $C_{(ij)(k)(l)}$ means a collineation interchanging $i$ and $j$ and leaving $k$ and $l$ fixed. Then, in the first place, there is a set of four reflexions (including identity) in the reference triangle, i. e. leaving the vertices of the reference triangle for centers and the sides for axes. In the notation just explained they are:

Firstly, $\quad C_{(12)(34)},\ C_{(13)(24)},$ and $C_{(14)(23)}$.

These are of the type

$$x_0' = x_0, \quad x_1' = x_1, \quad x_2' = -x_2.$$

Secondly, there are reflexions

$$C_{(34)(1)(2)}, \ C_{(14)(2)(3)}, \ C_{(13)(2)(4)}, \ C_{(24)(1)(3)}, \ C_{(12)(3)(4)}, \ C_{(13)(2)(4)},$$

which interchange two of the points and leave the other two fixed. E. g.

$$x_0' = x_1, \quad x_1' = x_0, \quad x_2' = x_2,$$

the center of which is $(1, -1, 0,)$ and the axis $x_0 - x_1 = 0$.

Thirdly, eight collineations of period three, leaving one point fixed and interchanging the other three cyclically

$$C_{(123)(4)}, \ C_{(132)(4)}, \ C_{(134)(2)}, \ C_{(143)(2)}, \ C_{(124)(3)}, \ C_{(142)(3)}, \ C_{(234)(1)}, \ C_{(243)(1)}.$$

To illustrate, take the transformation

$$x_0' = x_2, \quad x_1' = x_0, \quad x_2' = x_1,$$

which is a member of the set.

Lastly, six collineations of period four, interchanging the four points cyclically

$$C_{(1234)}, \ C_{(1324)}, \ C_{(1342)}, \ C_{(1243)}, \ C_{(1423)}, \ C_{(1432)}.$$

E. g. $x_0' = -x_2$, $x_1' = -x_1$, $x_2' = x_0$, is a member of this set.

The twenty-four collineations form a $G_{24}$, under which the four-point is invariant. Consider now the effects of these transformations on any conic passing through the four points $(1, \pm 1, \pm 1)$. Such a conic may be taken in the form

$$a_0 x_0^2 + a_1 x_1^2 + a_2 x_2^2 = 0$$

provided that $$(a) = 0.$$

The reflexions 1, $C_{(ij)(kl)}$ involve only a change of signs and will, therefore, leave the conic unaltered. The other elements of the $G_{24}$ in general send a conic on the four points into some other member of the pencil of conics determined by the four-point. Given, then, two conics considered as a quartic curve, the problem is to find correlations, and in particular, polarities, which will send any point of the curve into a line of the curve and conversely. This may be accomplished in either of two ways:

1°. The two conics may be interchanged under the correlations;

2°. Each conic may be unaltered.

## § 8. *General Case.*

Consider, first, the general case of two conics between which there exists no special relation. In general it is not possible to find a polarity that will leave each conic unaltered; but two conics related in a special manner do admit of such polarities and will be considered in the next article. It is necessary to look, then, for polarities that interchange the two conics.

Let

$$\alpha_0 x_0^2 + \alpha_1 x_1^2 + \alpha_2 x_2^2 = 0, \tag{31}$$

and

$$\beta_0 x_0^2 + \beta_1 x_1^2 + \beta_2 x_2^2 = 0, \tag{32}$$

where $(\alpha) = (\beta) = 0$, be two conics on the four points $(1, \pm 1, \pm 1)$. One of these is to be the polar reciprocal of the other as to some base conic. The latter may be taken in the form

$$a x_0^2 + b x_1^2 + c x_2^2 = 0; \tag{33}$$

since any polarity interchanging (31) and (32) will leave their common self-polar triangle unaltered and hence, either the latter is self-conjugate with respect to the base conic or else the base conic is tangent to two sides of the triangle, the third side being the chord joining the contacts. The latter possibility is considered in the next article. Now, if we require that the polarity as to (33) interchange (31) and (32) then the constants $a$, $b$, and $c$ are determined to be

$$a : b : c := \pm\sqrt{\alpha_0\beta_0} \; : \; \pm\sqrt{\alpha_1\beta_1} \; : \; \pm\sqrt{\alpha_2\beta_2}$$

Take all possible combinations of signs and we obtain the following four polarities:

$$\begin{array}{llll}
\quad\pi_0 & \quad\pi_1 & \quad\pi_2 & \quad\pi_3 \\
\xi_0 = \sqrt{\alpha_0\beta_0}\,x_0, & = -\sqrt{\alpha_0\beta_0}\,x_0, & = \sqrt{\alpha_0\beta_0}\,x_0, & = \sqrt{\alpha_0\beta_0}\,x_0, \\
\xi_1 = \sqrt{\alpha_1\beta_1}\,x_1, & = \sqrt{\alpha_1\beta_1}\,x_1, & = -\sqrt{\alpha_1\beta_1}\,x_1, & = \sqrt{\alpha_1\beta_1}\,x_1, \\
\xi_2 = \sqrt{\alpha_2\beta_2}\,x_2, & = \sqrt{\alpha_2\beta_2}\,x_2, & = \sqrt{\alpha_2\beta_2}\,x_2, & = -\sqrt{\alpha_2\beta_2}\,x_2,
\end{array}$$

One of these, say $\pi_0$, having been obtained as above, then the others are given by the products $\pi_0 \cdot C_{(ij)(kl)}$. These are evidently correlations and they interchange the two conics, since $\pi_0$ interchanges them and $C_{(ij)(kl)}$ leave them fixed. That they are polarities appears from the fact that the products $\pi_0 C_{(mn)(pq)} \cdot \pi_0 C_{(rs)(tv)}$ which are collineations leaving each conic unaltered, must be contained in the set $C_{(ij)(kl)}$, and in particular $\pi_0 C_{(ij)(kl)} \cdot \pi_0 C_{(ij)(kl)}$ must be identity. Furthermore no other correlation could exist which would

transform the one conic into the other. For supposing such a one to exist, say $\pi_m$, then the product $\pi_0 \cdot \pi_m$ must be contained in the set $C_{(ij)(kl)}$, i. e.

$$\pi_0 \cdot \pi_m = C_{(mn)(pq)}$$

But

$$\pi_0 \cdot [\pi_0 \cdot C_{(mn)(pq)}] = C_{(mn)(pq)}$$

And hence

$$\pi_m = \pi_0 \cdot C_{(mn)(pq)}.^*$$

Hence the number of polarities is exhausted.

That there are just four polarities which interchange the conics is evident from a geometrical point of view. For, the four points of intersection of the conics must go into their common lines. It would appear then that twenty-four polarities are possible; but any polarity sending one of the four common points into one of the four common tangents carries a unique transformation of the three other points into the three other tangents. Hence there are only four polarities possible.†

The elements 1, $C_{(ij)(kl)}$ form a $G_4$ of collineations under which the quartic is invariant. Furthermore it was shown above that all correlations leaving the curve unaltered are included in the set $\pi_0$, $[\pi_0 \cdot C_{(ij)(kl)}]$ and that the products of the latter, two at a time give $C_{(ij)(kl)}$.

Hence the Theorem: *A quartic curve, composed of two conics, is invariant under a $G_8$, consisting of four collineations and four correlations.*

## § 9. *Two Conics Subject to the Condition* $\Delta\theta_1^3 = \Delta_1\theta^3$.

We come next to the case of two conics admitting not only the polarities of the preceding article but also a second kind, viz. those which leave each conic separately unaltered.

Assume a pair of conics on the four points $(1, \pm 1, \pm 1)$, and such that either one of them is reflected into the other by one of the collineations $C_{(ij)(kl)}$,—say by the collineation

$$x_0' = x_0, \quad x_1' = x_2, \quad x_2' = x_1,$$

Such a pair are

$$a_0x_0^2 + a_1x_1^2 + a_2x_2^2 = 0, \tag{34}$$

$$a_0x_0^2 + a_2x_1^2 + a_1x_2^2 = 0. \tag{35}$$

---

* See Weber, Lehrbuch der Algebra, Vol. II, p. 4.

† On relations between two conics see Clebsch, Léçons sur La Géométrie, Vol. I, p. 150 *et seq.*

Furthermore, since

$$C_{(lm)(pq)} \cdot C_{(lm)(p)(q)} = C_{(pq)(l)(m)},$$

then the conics will be reflected into each other by a second member of the set $C_{(ij)(k)(l)}$.

Just as in the preceding case the curve is unaltered by the group 1, $C_{(ij)(kl)}$, $\pi_0$, $\pi_0 \cdot C_{(ij)(kl)}$, the polarities being as follows:

$$\begin{array}{llll}
\quad\pi_0 & \quad\pi_1 & \quad\pi_2 & \quad\pi_3 \\
\xi_0 = a_0 x_0, & = -a_0 x_0, & = a_0 x_0, & = a_0 x_0, \\
\xi_1 = \sqrt{a_1 a_2}\, x_1, & = \sqrt{a_1 a_2}\, x_1, & = -\sqrt{a_1 a_2}\, x_1, & = \sqrt{a_1 a_2}\, x_2, \\
\xi_2 = \sqrt{a_1 a_2}\, x_2, & = \sqrt{a_1 a_2}\, x_2, & = \sqrt{a_1 a_2}\, x_2, & = -\sqrt{a_1 a_2}\, x_2.
\end{array}$$

Now the reflexions $C_{(ij)(k)(l)}$ have the effect of interchanging the two conics, and $C_{(ij)(kl)}$ leave each unaltered. Hence the effect of the products $C_{(ij)(k)(l)} \cdot C_{(ij)(kl)}$ is to interchange the two. By these products we obtain four collineations which send each conic into the other. The four include the two reflexions $C_{(ij)(k)(l)}$ and two of the set $C_{(ijkl)}$, i. e. collineations of period four. The eight elements form a collineation $G_8$ under which the curve is invariant. Add now a polarity $\pi_0$ which interchanges the pair of conics. The products $\pi_0 \cdot C_{(ij)(kl)}$, as before, interchange the pair of conics; but the products $\pi_0 \cdot C_{(ij)(k)(l)}$ and $\pi_0 \cdot C_{(ijkl)}$, leave each conic fixed. Of these the first two are polarities and the other two correlations of period four. The four correlations are as follows:

$$\begin{array}{llll}
\multicolumn{2}{c}{\underline{\quad\pi_0 \cdot C_{(ij)(k)(l)}\quad}} & \multicolumn{2}{c}{\underline{\quad\pi_0 \cdot C_{(ijkl)},\quad}} \\
\xi_0 = a_0 x_0, & = -a_0 x_0, & = a_0 x_0, & = a_0 x_0, \\
\xi_1 = \sqrt{a_1 a_2}\, x_1, & = \sqrt{a_1 a_2}\, x_2, & = -\sqrt{a_1 a_2}\, x_2, & = \sqrt{a_1 a_2}\, x_2, \\
\xi_2 = \sqrt{a_1 a_2}\, x_2, & = \sqrt{a_1 a_2}\, x_1, & = \sqrt{a_1 a_2}\, x_1, & = -\sqrt{a_1 a_2}\, x_1.
\end{array}$$

It is readily seen that the determinants of the first two are symmetrical and that those of the other two are not. Hence the former are polarities while the latter are correlations. That the correlations are of period four may be easily verified and is evident from the fact that they are $\pi_0 \cdot C_{(ijkl)}$ which, raised to the fourth power, is $\pi_0^4 \cdot C^4_{(ijkl)}$, i. e. identity. Furthermore there can be no other correlations leaving the pair of conics unaltered, a fact easily proved just as in the preceding case. Hence the curve is invariant under a $G_{16}$ of collineations and correlations. Of the latter six are polarities and two are of period four.

Consider now the base conics of the polarities $\pi_0 \cdot C_{(ij)(k)(l)}$. Their equations are

$$a_0 x_0^2 \pm 2\sqrt{a_1 a_2}\; x_1 x_2 = 0. \tag{36}$$

These are of the nature of conjugate hyperbolas and may be called conjugate conics. Furthermore each of the original pair is conjugate to both of (36). What we have then is this: Two conics (31) and (32) which admit of a reflexion, the one into the other; two conics (36) each of which is conjugate to the other and is also conjugate to both (31) and (32).

We inquire now as to whether or not it was necessary to assume conics that admit of one of the reflexions $C_{(ij)(k)(l)}$ in order to obtain polarities whose base conics are conjugate to the original pair. Two conics admitting a reflexion may be taken in the forms

$$ax_0^2 + bx_1^2 + cx_2^2 \pm 2gx_0x_2 = 0.$$

The invariants, using Salmon's notation (Conic Sections, p. 334), are as follows:

$$\Delta = abc - bg^2, \quad \Delta_1 = abc - bg^2, \quad \theta = (3abc - bg^2), \quad \theta_1 = (3abc - bg^2).$$

Since an invariant relation must be homogeneous and of the same degree in the coefficients of both forms, the only relation subsisting between the invariants is

$$\Delta\theta_1^3 = \Delta_1\theta^3.$$

We seek next the condition that two conics have a common conjugate conic and may admit, therefore, of polarities such as $\pi_0 \cdot C_{(ij)(k)(l)}$. The pencil of conics having double contact with $(x^2) = 0$, and with $(x\xi) = 0$ as the common chord of contact is

$$(x\xi)^2 - \lambda[(x^2)(\xi^2) - (x\xi)^2] = 0.^* \tag{37}$$

Values of $\lambda$ that are equal but of opposite signs give conjugate pairs. For $\lambda = \pm 1$ we have

$$(x^2)(\xi^2) - 2(x\xi)^2 = 0 \tag{38}$$

and

$$(x^2) = 0 \tag{39}$$

Now a second line, $(x\eta) = 0,$

will determine another pencil

$$(x\eta)^2 - \mu[(x^2)(\eta^2) - (x\eta)^2] = 0 \tag{40}$$

and for $\mu = \pm 1$ the conjugate pair

$$(x^2)(\eta^2) - 2(x\eta)^2 = 0, \tag{41}$$

* See Salmon's Conic Sections, p. 340.

$$(x^2) = 0. \tag{42}$$

Hence (38) and (41) are both conjugate to $(x^2) = 0$. Writing $\sigma_1 = (\xi^2)$ and $\sigma_2 = (\eta^2)$, the invariants are

$$\Delta = -\sigma_1{}^3, \quad \Delta_1 = -\sigma_2{}^3, \quad \theta = \sigma_1\lambda, \quad \theta_1 = \sigma_2\lambda,$$

where

$$\lambda = [\sigma_1\sigma_2 - 4(\xi_0\eta_0 + \xi_1\eta_1 + \xi_2\eta_2)^2].$$

The only invariantive relation subsisting between them is

$$\Delta\theta_1{}^3 = \Delta_1\theta^3.$$

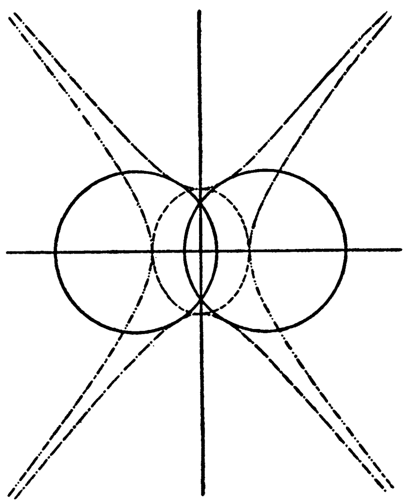

FIG. 3.

That is, the invariant relation that two conics admit of a reflexion the one into the other is identically the same as the condition that the two have a common conjugate conic. Hence the necessary and sufficient condition that polarities of the type $\pi_0 \cdot C_{(ij)(k)(l)}$ exist, is that the pair of conics admit of a reflexion and is expressed analytically by the above relation between the invariants.

We summarize this case by the Theorem: *Two conics characterized by the relation $\Delta\theta_1{}^3 = \Delta_1\theta^3$ are invariant under a $G_{16}$ consisting of collineations and correlations. Of the latter six are polarities and two are of period four.*

In Fig. 3 the circles are the original pair of conics admitting a reflexion. Only three polarities are real, as shown in the figure.

## §10. *The Clebschian Pair.*

We come finally to a special pair of conics which is invariant under the entire $G_{24}$ of collineations and therefore, by adding a polarity and a $G_{48}$ of collineations and correlations.

Suppose the conics of the preceding case are required to admit not only of two members of the set $C_{(ij)(k)(l)}$, as in that case, but also of a third member of the same set of collineations. Putting this further condition in the two conics we obtain the pair

$$x_0^2 + \omega x_1^2 + \omega^2 x_2^2 = 0, \quad x_0^2 + \omega^2 x_1^2 + \omega x_2^2 = 0. \tag{43}$$

These are either interchanged or left separately unaltered by the entire $G_{24}$ of collineations. Hence, by adding a polarity, they are unaltered by a $G_{48}$ of correlations and collineations. Of the former ten are polarities. Hence the Theorem: *The Clebschian pair of conics is invariant under a $G_{48}$ of collineations and correlations. Of the latter ten are polarities and the others of periods three and four.*